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**CALCULATION OF TURBULENT BOUNDARY
LAYERS WITH HEAT TRANSFER AND
PRESSURE GRADIENT UTILIZING A
COMPRESSIBILITY TRANSFORMATION**

**Part II - Constant Property Turbulent Boundary-Layer
Flow With Simultaneous Mass Transfer
and Pressure Gradient**

by J. Boccio and C. Economos

Prepared by

GENERAL APPLIED SCIENCE LABORATORIES, INC.

Westbury, L. I., New York 11590

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16. Abstract <p>An analysis of the incompressible turbulent boundary layer, developing under the combined effects of mass transfer and pressure gradient, is presented in this paper. A strip-integral method is employed whereby two of the three governing equations are obtained by integrating the combined momentum and continuity equation to 50 percent and 100 percent, respectively, of the boundary-layer height. The latter equation is the usual momentum-integral equation; the former equation requires specification of shear at the point $\bar{\eta}=\bar{\eta}^* = 0.5$. Accordingly, Clauser's equilibrium eddy-viscosity law is assumed valid at this point. The third and final equation is obtained by specifying that Stevenson's velocity profiles apply throughout the domain of interest, from which a skin-friction law can be derived.</p> <p>Comparisons of the numerical results with the experiments of McQuaid, which include combined effects of variable pressure gradient and mass transfer, show good agreement.</p>					
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FOREWORD

The present report is one of a series of three reports describing a new computer program which predicts turbulent boundary-layer behavior for a condition involving both heat transfer and pressure gradient. Part I serves as a summary report and describes the general analysis which is utilized in the numerical calculations scheme. In Part II, the requisite low speed formulation consisting of a constant property flow with combined pressure gradient and mass transfer is described. Part III describes the numerical and computational procedures involved and serves as a computer program manual.

The titles in the series are:

- PART I - Summary Report - "Calculation of Turbulent Boundary Layers with Heat Transfer and Pressure Gradient Utilizing a Compressibility Transformation," by C. Economos and J. Boccio.
- PART II - "Constant Property Turbulent Boundary-Layer Flow with Simultaneous Mass Transfer and Pressure Gradient," by J. Boccio and C. Economos.
- PART III - "Computer Program Manual," by J. Schneider and J. Boccio.

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CONTENTS

	Page
SUMMARY	vii
INTRODUCTION	1
SYMBOLS	5
ANALYSIS	8
A. Basic Equations	8
B. Velocity Profile Representation	10
C. The Skin Friction Law	16
D. Final Working Forms of the Governing Equations	17
E. Initial and Boundary Conditions	19
RESULTS AND DISCUSSIONS	21
CONCLUSIONS	28
APPENDIX	29
REFERENCES	35

SUMMARY

An analysis of the incompressible turbulent boundary layer, developing under the combined effects of mass transfer and pressure gradient, is presented in this paper. A strip-integral method is employed whereby two of the three governing equations are obtained by integrating the combined momentum and continuity equation to 50% and 100%, respectively, of the boundary-layer height. The latter equation is the usual momentum-integral equation; the former equation requires specification of shear at the point $\bar{\eta} = \bar{\eta}^* = 0.5$. Accordingly, Clauser's equilibrium eddy-viscosity law is assumed valid at this point. The third and final equation is obtained by specifying that Stevenson's velocity profiles apply throughout the domain of interest, from which a skin-friction law can be derived.

Comparisons of the numerical results with the experiments of McQuaid, which include combined effects of variable pressure gradient and mass transfer, show good agreement.

CALCULATION OF TURBULENT BOUNDARY LAYERS WITH
HEAT TRANSFER AND PRESSURE GRADIENT UTILIZING A
COMPRESSIBILITY TRANSFORMATION

PART II - CONSTANT PROPERTY TURBULENT BOUNDARY -
LAYER FLOW WITH SIMULTANEOUS MASS TRANSFER AND
PRESSURE GRADIENT

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INTRODUCTION

In recent years there has been an increasingly greater interest in turbulent boundary layers. On one hand, continuing research has been aimed toward finding an understanding of turbulent shear flows of which the boundary layer is probably the most interesting example. On the other hand, an insistent demand still exists for reliable prediction methods to calculate stream-wise development of gross properties of the boundary layer as well as its mean velocity and shear-stress distribution.

The serious interest in turbulent boundary layers has been evidenced by the fact that relatively large symposiums were dedicated to the subject. In 1968, two meetings were held - both of which served largely pragmatic interests. One held at Stanford University (Reference 1) dealt with the study of incompressible turbulent boundary layers, while the study of its compressible counterpart was the primary concern at the NASA sponsored meeting held at Langley Field, Virginia (Reference 2). The 1968 Stanford Conference revealed many interesting aspects of the technical problem associated with the study of incompressible turbulent flows over impermeable surfaces subjected to both zero and non-zero pressure gradients. One outcome was the fact that some integral methods, specifically the dissipation-integral method of Alber (Reference 3) and the strip-integral method of Moses (Reference 4) were just as successful in predicting the mandatory test cases posed as the more elaborate differential or mean-field methods of Mellor and Herring (Reference 5), Cebeci and Smith (Reference 6) and Beckwith and Bushnell (Reference 7).

At the Langley conference, the published theoretical predictions by Economos (Reference 8) dealing with the study of compressible turbulent boundary layers over flat plates with mass addition and, more recently, the theoretical predictions made by Economos and Boccio (Reference 9) that treat the problem of compressible boundary-layer development with pressure gradient and heat transfer but no mass-transfer effects by utilizing a Coles'-type (Reference 10) compressibility transformation has revived, to some extent, the use of such types of transformations in the study of turbulent boundary-layer flows.

As discussed in Part I of this report, reexamination of this compressibility transformation in an effort to improve velocity profile representation, has led to the development of a somewhat more general form of transformation. This new form relates the compressible boundary-layer flow under the influence of pressure gradient and heat transfer to an incompressible one in which pressure gradient and mass transfer occur simultaneously. Thus, in order to achieve the objectives of this contract, the first step was development of a formulation which described the latter flow. This report will describe such a formulation.

An integral method of approach has been chosen because of its virtue of incorporating these effects on the turbulence structure in an implicit and global manner. Thus, the avoidance of local turbulence assumptions that are required in field methods, the fact that there is a dearth of corroborative experimental information which relates the dependence of mass transfer on eddy viscosity, and the simplicity of the method, all go into making the integral approach an attractive calculation tool for this endeavor.

Accordingly, at the outset, a two-parameter profile representation is deemed necessary. And because of its firmer theoretical basis, the Stevenson defect law (Reference 11) has been chosen over those postulated by Mickley and Smith (Reference 12) and McQuaid (Reference 13). The analysis could have used the defect law posed by Simpson (Reference 14) which is considered to be a logical extension to Coles "law of the wall - law of the wake formulation" to account for the non-constancy of the von Karman parameter, k_1 , which occurs at lower momentum thickness Reynolds number (< 6000), but its added complication and only slight improvement did not mandate its use. From the defect

velocity representation emerges a skin-friction law from which a differential equation can be obtained relating the requisite three dependent variables that define the system; namely, the Coles wake parameter, π , the wall skin-friction parameter, \bar{C}_f , and the Reynolds number based upon boundary-layer height, \bar{R} . Consequently, in addition to this equation and the usual momentum-integral equation, a third differential equation is required to close the system. In this context, several approaches are available based upon their success reported at the 1968 Stanford conference. For example, higher moment equations such as the moment of momentum equation as Alber uses for the non-transpired problem can be utilized. And indeed, this has been the case in the method of Lubard and Fernandez (Reference 15) who have treated the identical problem as that presented herein. However, to implement this method they must extend the procedure of Alber by not only uncoupling the attendant dissipation integral from the local pressure gradient, but they must also decouple its dependence from the mass transfer rate. Consequently, in addition to using Alber's empirical curve fit which relates the Clauser shape parameter with pressure gradient parameter and which they have to generalize to account for mass transfer, they must further assume special forms for the variation of mass transfer rate to associate with an "equilibrium" flow in order to close their system. Although the numerical examples cited are found to be weakly dependent upon which variation (three in all) is used and although they present rather good agreement with experiments, it is felt that use of such an assumption could be restrictive insofar as incorporating it into other types of pressure gradient and mass transfer rate distributions.

Accordingly, the approach taken herein to obtain the remaining governing equation and which is felt to be one degree less empirical with regard to its extension to problems involving transpiration is the strip-integral method of Moses - a method which has also enjoyed reasonable success at the Stanford conference. This additional equation is obtained by integrating the momentum equation up to 50% of the boundary-layer height, at which point the shear stress is a priori assumed to be that postulated by Clauser (Reference 16) for equilibrium flows. Thus, an assumption based upon equilibrium flows extended to account for mass transfer and arbitrary pressure gradient is required here as well as in the method of Reference 15. However, this assumption is only required at one point in the boundary layer and not across the entire boundary layer as is required

in the aforementioned reference. Also, no empirical curve fit of non-transpired boundary-layer data a priori extended to the transpired problem need be made.

In the following sections, the integral method is developed for the two-dimensional case and for the case where the injectant is the same fluid as the external stream. Numerical results are obtained for three specific problems, namely, those termed by McQuaid (Reference 17) as pressure distribution I, II, III.

SYMBOLS

A_i	elements of ordinary differential equations, c.f. Appendix and Eq. (38)
$A(\bar{\phi}, \tilde{v}_w)$	functional form, c.f. Eq. (24)
$B(\tilde{v}_w)$	$(1/4)\tilde{v}_w$
c_j	column vector relating to right-hand-side of the system of ordinary differential equations c.f. Appendix and Equation (38)
\bar{c}_f	local skin friction coefficient, $2\bar{\tau}_w/\bar{\rho}_e \bar{u}_e^2$
$F_i(\bar{\eta}^*) ; i = 1, 2, 3$	functional forms, c.f., Eqs. (A-8), (A-9), (A-10)
$G(\pi, \bar{\eta})$	functional form, c.f. Eq. (23)
\bar{H}	form factor
$I(\bar{\eta}, p, q)$	definite integrals described by Eq. (A-4)
$J(\pi)$	functional form, c.f. Eq. (26)
K	$-(1/k_1)$
k_1	constant, 0.4115
k_2	constant, 4.9
\bar{p}_e	external pressure, lb/ft ²
$Q_i(\pi, \bar{\eta}) ; i = 1, \dots, 5$	functional forms, c.f. Eq. (A-5)
\bar{R}	Reynolds number based upon boundary-layer height, $(\bar{\rho} \bar{u}_{e,o} \bar{\delta}/\bar{\mu})$
R_y	Reynolds number based upon local value of \bar{y} , $(\bar{\rho} \bar{u}_{e,o} \bar{y}/\bar{\mu})$
U	generalized velocity function, c.f. Eq. (8)

\bar{u}_e	$\bar{u}_e/\bar{u}_{e,o}$
$\bar{u}_\tau, \bar{u}_{\tau w}$	shear velocities, $(\bar{\tau}/\bar{\rho})^{\frac{1}{2}}$, $(\bar{\tau}_w/\bar{\rho})^{\frac{1}{2}}$, respectively, ft/sec
\bar{u}, \bar{v}	velocities respectively along and normal to body, ft/sec
\bar{u}^+, \bar{v}^+	velocity ratios $\bar{u}/\bar{u}_{\tau w}$; $\bar{v}/\bar{u}_{\tau w}$, respectively
\bar{v}_w	$\bar{v}_w/\bar{u}_{e,o}$
$w(\bar{\eta})$	difference between outer edge value and local value of Coles wake function, c.f. Eq. (19)
$w(\bar{\eta})$	Coles wake function
$w_i(\bar{\eta})$; $i = 1, 2, 3$	various curve fits to Coles wake function, c.f. Eq. (21)
\bar{x}, \bar{y}	space coordinates respectively along and normal to body
\bar{y}^+	$\bar{\rho}\bar{u}_{\tau w} \bar{y}/\bar{\mu}$
\bar{y}_s^+	value of \bar{y}^+ at laminar sub-layer height
β	experimental constant (0.018) in the description of eddy viscosity c.f. Eq. (6)
$\bar{\delta}$	boundary layer height, ft
$\bar{\delta}^*$	displacement thickness, ft
$\bar{\epsilon}$	eddy viscosity, ft^2/sec
$\bar{\epsilon}^+$	viscosity ratio, $\bar{\rho}\bar{\epsilon}/\bar{\mu}$
$\bar{\eta}$	$\bar{y}/\bar{\delta}$
$\bar{\eta}^*$	a particular value of $\bar{\eta}$ taken within to be equal to 0.5
$\bar{\theta}$	momentum thickness, ft
$\Lambda_1^{(\bar{\eta})}$ $\Lambda_2^{(\bar{\eta})}$	definite integrals, c.f. Eq. (31), (30), respectively

$\bar{\mu}$	molecular viscosity, lb sec/ft ²
$\bar{\nu}$	kinematic viscosity, ft ² /sec
π	Coles wake parameter
$\bar{\rho}$	density, lb sec ² /ft ⁴
Σ	$\bar{\theta}/\bar{\delta}$
$\bar{\tau}$	shear stress, lb/ft ²
$\bar{\phi}$	skin-friction parameter, $\bar{u}_e/\bar{u}_{\tau w}$
$\bar{\chi}$	Reynolds number defined as $\bar{u}_{e,o}(\bar{x}-\bar{x}_o)/\bar{\nu}$
Ω	$\bar{\delta}^*/\bar{\delta}$

Subscripts

$()_e$	pertains to local condition at edge of boundary layer
$()_o$	pertains to initial conditions
$()_s$	pertains to laminar sub-layer height
$()_w$	pertains to condition at wall

Superscripts

(\sim)	normalization with respect to external value unless otherwise noted, i.e., $\tilde{u} = \bar{u}/\bar{u}_e$
$()^*$	designates properties at $\bar{\eta}^*$
$()^{(\bar{\eta}^*)}$	designates integration performed between the couple $(0, \bar{\eta}^*)$
$()^{(1)}$	designates integration performed between couple $(0, 1)$
$()'$	$d/d\bar{\chi}$

ANALYSIS

A. Basic Equations

The differential equations describing the mean flow of an incompressible turbulent boundary layer for a two-dimensional flow field are taken to be

Continuity:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \quad (1)$$

Momentum:

$$\bar{\rho} \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{\rho} \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = - \frac{d\bar{p}_e}{d\bar{x}} + \frac{\partial \bar{\tau}}{\partial \bar{y}} \quad (2)$$

Shear Stress:

$$\bar{\tau} = \bar{\mu} \frac{\partial \bar{u}}{\partial \bar{y}} - \bar{\rho} \bar{u}'\bar{v}' = (\bar{\mu} + \bar{\rho}\epsilon) \frac{\partial \bar{u}}{\partial \bar{y}} \quad (3)$$

External Flow:

$$\frac{d\bar{p}_e}{d\bar{x}} = -\bar{\rho}_e \bar{u}_e \frac{d\bar{u}_e}{d\bar{x}} \quad (4)$$

The developing flow-field is subject to the boundary conditions

$$\bar{u} = 0 \quad \bar{v} = \bar{v}_w(\bar{x}) \quad \text{at } \bar{y} = 0$$

$$\bar{u} = \bar{u}_e(\bar{x}) \quad \text{at } \bar{y} \rightarrow \infty$$

A set of ordinary differential equations can be obtained from the boundary-layer equations (1) and (2) by integrating in \bar{y} yielding after some manipulations

$$\begin{aligned}
& \frac{d}{d\bar{x}} \int_0^{\bar{\eta}^*} \bar{u}^2 d\bar{\eta} - \bar{u}^* \frac{d}{d\bar{x}} \int_0^{\bar{\eta}^*} \bar{u} d\bar{\eta} - \frac{\bar{v}_w \bar{u}^*}{\bar{R}} + \frac{d}{d\bar{x}} (\ln \bar{R}) \left\{ \int_0^{\bar{\eta}^*} \bar{u}^2 d\bar{\eta} - \bar{u} \int_0^{\bar{\eta}^*} \bar{u} d\bar{\eta} \right\} \\
& + \frac{d}{d\bar{x}} (\ln \bar{u}_e) \left\{ \int_0^{\bar{\eta}^*} 2\bar{u}^2 d\bar{\eta} - \bar{u}^* \int_0^{\bar{\eta}^*} \bar{u} d\bar{\eta} - \bar{\eta}^* \right\} = \frac{1}{\bar{R}} \left\{ \bar{\tau}^* - \frac{1}{\bar{\phi}^2} \right\}
\end{aligned}
\tag{5}$$

where the superscript (\sim) denotes, unless otherwise noted, that the quantity has been ratioed with the local edge velocity, \bar{u}_e^* . In addition, the following definitions are required, namely:

$$\begin{aligned}
\bar{x} & \equiv \bar{u}_{e,o} (\bar{x} - \bar{x}_o) / \bar{v} \\
\bar{\eta} & \equiv \bar{y} / \bar{\delta} \\
\bar{R} & \equiv \bar{u}_{e,o} \bar{\delta} / \bar{v} \\
\bar{u}_e & \equiv \bar{u}_e / \bar{u}_{e,o} \\
\bar{\phi} & \equiv \bar{u}_e / (\bar{\tau}_w / \bar{\rho})^{1/2} \\
\bar{\tau} & \equiv \bar{\tau} / \bar{\rho}_e \bar{u}_e^2 \\
()_o & \equiv \text{initial quantities} \\
()^* & \equiv \text{properties at } \bar{\eta} = \bar{\eta}^* \\
()_e & \equiv \text{external conditions} \\
()_w & \equiv \text{wall conditions}
\end{aligned}$$

* To be consistent with Parts I and III of this contractual requirement, superscript (\sim) connotes properties in the incompressible plane.

Any number of differential equations can be obtained by evaluating Equation (5) at different values of $\bar{\eta}^*$. Evaluation at $\bar{\eta}^* = 1$ results in the usual momentum-integral equation. Accordingly, with two unknown parameters assumed for the velocity profile representation (which is to be discussed later), only one additional equation and choice of $\bar{\eta}^*$ is required. Values of $\bar{\eta}^*$ between 0.2 to 0.5 have been tried by Moses and by Baronti (Reference 18) with little effect on the final results. Thus, the arguments posed by these two authors are extended here by considering as the second governing equation that which results by letting $\bar{\eta}^* = 0.5$, at which point Clauser's eddy viscosity law is also assumed to hold, i.e.

$$\bar{\epsilon}^* = \beta \bar{u}_e \bar{\delta}^* = 0.018 \bar{u}_e \bar{\delta}^* \quad (6)$$

Considering the molecular viscosity negligible at $\bar{\eta}^* = 0.5$ then the shear stress becomes

$$\bar{\tau}^* = (\bar{\rho} \bar{\epsilon} \frac{\partial \bar{u}}{\partial \bar{y}})^* = \beta \bar{\Omega} (\frac{\partial \bar{u}}{\partial \bar{\eta}})^* \quad (7)$$

where the normalized displacement thickness is

$$\bar{\Omega} \equiv \bar{\delta}^* / \bar{\delta}$$

B. Velocity Profile Representation

The turbulent boundary layer is generally assumed to consist of three distinct flow regions: (1) the viscous sublayer, (2) the turbulent core, and (3) the wake region. A fourth region termed as a buffer or blending region which is bracketed by regions (1) and (2) above must also be considered in the description of a turbulent boundary layer. However, buffer region expressions are not as well agreed on and normally any expression for this region incorporates the sublayer or the sublayer and turbulent core (Kleinstein, Reference 19).

In the viscous sublayer, molecular viscosity dominates. In the buffer region, transition to the fully turbulent part of the boundary layer takes place and the laminar and turbulent mechanism appear to be of equal importance. In the turbulent core, molecular viscosity is assumed to completely lose its significance and the velocity profile depends only on the turbulence. Beyond the turbulent core is the wake region which is essentially another transition region in that the turbulence becomes intermittent and the flow changes from fully turbulent to the free stream condition.

In incompressible flow over impermeable flat plates, various "laws" have been used to describe these regions. Regions (1), (2) and the buffer region, which is not considered any further in this report, are referred to as the "law of the wall" region. The wake region, so called because of the flow similarity to that of a wake propagating into an otherwise undisturbed flow, has been described by the "law of the wake" devised by Coles (Reference 20).

The extension of these laws to describe the velocity profiles over permeable bodies are discussed in this section. The formulation is brief and represents a summary of existing work due to several contributors; additional discussions on this subject can be found in Reference 21.

A three-layer model of the boundary layer is considered and within this framework, the choice of the profiles is believed to be the best representation available.

A generalized velocity function defined as

$$U = \int^{u^+} d\bar{u}^+ / \bar{u}_\tau^+ \quad (8)$$

is introduced where

$$\bar{u}^+ \equiv \bar{u} / \bar{u}_{\tau w} = \bar{u} \phi$$

$$\bar{u}_{\tau w} \equiv (\bar{\tau}_w / \bar{\rho})^{1/2}$$

$$\bar{u}_\tau^+ \equiv \bar{u}_\tau / \bar{u}_{\tau w}$$

Now, in considering the ramifications associated with the definitions of the laminar sublayer and the law of the wall, namely, that within these regions the streamwise gradients do not significantly affect the shear distribution, requires then that this generalized velocity function be

$$U = (2/\bar{v}_w^+) [(1 + \bar{v}_w^+ \bar{u}^+)^{\frac{1}{2}} - 1] \quad (9)$$

Furthermore, within the laminar sublayer where it is assumed that the quantity $\bar{\epsilon}^+$, i.e., $(\bar{\rho}\bar{\epsilon}/\bar{\mu})$ is considered much less than unity, it follows that the velocity distribution \bar{u}^+ of the form

$$\bar{u}^+ \bar{v}_w^+ = \exp(\bar{v}_w^+ \bar{y}^+) - 1 \quad 0 < \bar{y}^+ \leq \bar{y}_s^+ \quad (10)$$

where a measure of the laminar sublayer height, \bar{y}_s^+ , is, as yet, unknown and where

$$\begin{aligned} \bar{v}_w^+ &\equiv \bar{v}_w / \bar{u}_{\tau w} = \bar{v}_w \bar{\phi} \\ \bar{y}^+ &\equiv \bar{\rho} \bar{u}_{\tau w} \bar{y} / \bar{\mu} = \bar{R} \bar{U}_e \bar{\eta} / \bar{\phi} \end{aligned}$$

With regard to the law-of-the-wall region where $\bar{\epsilon}^+$ is considered to be much larger than unity then

$$\bar{\epsilon}^+ \frac{dU}{d\bar{y}^+} = \bar{u}_{\tau}^+ \quad (11)$$

Accordingly, by considering the momentum transport theory of Prandtl with its attendant mixing-length hypothesis, namely that

$$\bar{\epsilon}^+ = k_1 \bar{u}_{\tau}^+ \bar{y}^+ \quad (12)$$

yields, when substituted into the above equation and integrated,

$$U = k_2 + (1/k_1) \ln \bar{y}^+ \quad \bar{y}^+ \geq \bar{y}_s^+ \quad (13)$$

The constants k_2 and k_1 are taken to be those associated with the zero mass-transfer case. A posteriori justification for this assumption has been provided by experimental measurements for the zero pressure-gradient case with mass injection and is discussed in Reference 21. For pressure-gradient cases its justification will be provided by comparisons with the experiments.

Finally, by combining Equation (9) with Equation (13) there is obtained the law-of-the-wall equation with transpiration in the form

$$U = [(1 + \bar{v}_w^+ \bar{u}^+)^{\frac{1}{2}} - 1] = k_2 + (1/k_1) \ln \bar{y}^+ ; \quad \bar{y}^+ \geq \bar{y}_s^+ \quad (14)$$

which is identical to that deduced by Stevenson. To complete the profile formulation within these two regions, the value of \bar{y}_s^+ must be determined. Requiring the velocity to be continuous at this point requires the equality of Equation (10) and Equation (14) which is provided if

$$1 + (\bar{v}_w^+/2) [k_2 + (1/k_1) \ln \bar{y}_s^+] = \exp[(1/2) \bar{v}_w^+ \bar{y}_s^+] \quad (15)$$

With the velocity profiles determined within the inner turbulent region, the variation in the outer region, the so-called defect law, is readily derivable from a dimensional similarity argument. Thus, by considering the existence of a region of common validity between the law-of-the-wall and the law-of-the-wake and the fact that the latter law avoids a direct confrontation with the physical mechanism of shear turbulence requires the defect law to have the form

$$f(\bar{u}_e^+, \bar{v}_w^+) - f(\bar{u}^+, \bar{v}_w^+) = F_1(\bar{\eta}) \quad (16)$$

Identifying the function, f , with the generalized velocity function, U , then requires that $F(\bar{\eta})$ must be

$$F(\bar{\eta}) = (2/\bar{v}_w^+) [(1 + \bar{u}_e^+ \bar{v}_w^+)^{\frac{1}{2}} - (1 + \bar{u}^+ \bar{v}_w^+)^{\frac{1}{2}}] \quad (17)$$

and, by considering the universality of $F(\bar{\eta})$, then its representation can be taken from that which is associated with the impermeable wall problems. Thus, in the spirit of Coles, $F(\bar{\eta})$ behaves like

$$F(\bar{\eta}) = (\pi/k_1) W(\bar{\eta}) - (1/k_1) \ln \bar{\eta} \quad (18)$$

where

$$W(\bar{\eta}) \equiv w(1) - w(\bar{\eta}) \quad (19)$$

and $w(\bar{\eta})$ is the so-called Coles wake function. Thus, the defect law is obtained by equating Equation (17) with Equation (18), thereby yielding the general representation within the outer layer, i.e.,

$$U = (2/\bar{v}_w^+) [(1 + \bar{u}_w^+ \bar{u}^+)^{\frac{1}{2}} - 1] = k_2 + (1/k_1) \ln \bar{y}^+ + (\pi/k_1) w(\bar{\eta})$$

which further demonstrates the utility of the generalized velocity function U .

Thus, the velocity profile representation, taken in concert, is

Inner Region, law-of-the-wall region

$$\bar{u}^+ \bar{v}_w^+ = \exp(\bar{v}_w^+ \bar{y}^+) - 1 \quad 0 < \bar{y}^+ \leq \bar{y}_s^+ \quad (20a)$$

$$(2/\bar{v}_w^+) [(1 + \bar{u}^+ \bar{v}_w^+)^{\frac{1}{2}} - 1] = k_2 + (1/k_1) \ln \bar{y}^+ \quad \bar{y}^+ > \bar{y}_s^+ \quad (20b)$$

where \bar{y}_s^+ must satisfy the transcendental equation

$$1 + (\bar{v}_w^+/2) [k_2 + (1/k_1) \ln \bar{y}_s^+] = \exp[(1/2) \bar{v}_w^+ \bar{y}_s^+] \quad (20c)$$

Outer Region, law-of-the-wake region

$$(2/\bar{v}_w^+) [(1+\bar{v}_w^+ \bar{u}^+)^{1/2} - 1] = (1/k_1) \ln \bar{y}^+ + (\pi/k_1) w(\bar{\eta}) + k_2$$

$$\bar{\eta}_s < \bar{\eta} \leq 1 \quad (20d)$$

$$\bar{y}_s^+ < \bar{y}^+ \leq \bar{\delta}^+$$

Various curve fits to the Coles wake function have been incorporated into the numerical program, namely

$$w(\bar{\eta}) \equiv w_1(\bar{\eta}) = 2(3\bar{\eta}^2 - 2\bar{\eta}^3)$$

$$w(\bar{\eta}) \equiv w_2(\bar{\eta}) = 1 - \cos \pi \bar{\eta} \quad (21)$$

$$w(\bar{\eta}) \equiv w_3(\bar{\eta}) = 39\bar{\eta}^3 - 125\bar{\eta}^4 + 183\bar{\eta}^5 - 133\bar{\eta}^6 + 38\bar{\eta}^7$$

However, only the results using $w_1(\bar{\eta})$ are presented herein.

A working form of the velocity-profile representation is obtained from Equation (20d) by normalizing the velocity component with \bar{U}_e and the physical height with $\bar{\delta}$ yielding the following velocity-profile representation, which is considered to be more suitable for integration and differentiation.

Thus, the velocity profile is given by

$$\bar{u} = 1 + A(\bar{\varphi}, \bar{v}_w) G(\pi, \bar{\eta}) + B(\bar{v}_w) G^2(\pi, \bar{\eta}) \quad (22)$$

where

$$G(\pi, \bar{\eta}) \equiv [J(\pi) W(\bar{\eta}) + K \ln \bar{\eta}] \quad (23)$$

$$A(\bar{\varphi}, \bar{v}_w) \equiv -(1 + \bar{\varphi}^2 \bar{v}_w)^{1/2} (1/\bar{\varphi}) \quad (24)$$

$$B(\bar{v}_w) \equiv (1/4) \bar{v}_w \quad (25)$$

$$J(\pi) \equiv (\pi/k_1) \quad (26)$$

$$K \equiv -(1/k_1) \quad (27)$$

Note, Equation (22) is strictly applicable outside the laminar sublayer region; thus this region is neglected when this equation is substituted in the integrals associated with Equation (5).

C. The Skin Friction Law

A direct consequence of Equation (20d) is the skin-friction law obtained by placing $\bar{\eta}$ equal to unity at $\bar{y} = \bar{\delta}$. Slight manipulations yield

$$(1 + \bar{v}_w^+ \bar{\varphi})^{\frac{1}{2}} = 1 + (\bar{v}_w^+ / 2) [k_2 + (1/k_1) \ln \bar{\delta}^+ + (\pi/k_1) w(1)] \quad (28a)$$

which can be rewritten as

$$(1 + \bar{\varphi}^2 \bar{v}_w)^{\frac{1}{2}} = 1 + (\bar{v}_w \bar{\varphi} / 2) [k_2 + (1/k_1) \ln (\bar{R} \bar{U}_e / \bar{\varphi}) + (2\pi/k_1)] \quad (28b)$$

with, according to the curve fits of the wake function,

$$w(1) = 2$$

This equation, which relates the skin-friction parameter, $\bar{\varphi}$, a scaled boundary-layer thickness, \bar{R} , and the wake parameter, π , to the independent variable, \bar{x} , constitutes the remaining equation of the problem at hand. Implicit in the equation is the x -wise dependency of \bar{v}_w since \bar{v}_w and \bar{U}_e are considered to be known functions of \bar{x} . For later computational convenience differentiating Equation (28b) with respect to \bar{x} yields after considerable manipulations a working differential equation namely

$$\begin{aligned} & (\ln \bar{R})' (1/k_1) + (\ln \bar{U}_e)' (1/k_1) + (\pi)' (2/k_1) + (\ln \bar{\varphi})' \{ -(1/k_1) + \\ & + (2/\bar{\varphi} \bar{v}_w) [(1 + \bar{\varphi}^2 \bar{v}_w)^{\frac{1}{2}} - 1] - 2\bar{\varphi} (1 + \bar{\varphi}^2 \bar{v}_w)^{-\frac{1}{2}} \} + (\ln \bar{v}_w)' \{ (2/\bar{\varphi} \bar{v}_w) [(1 + \\ & \bar{\varphi}^2 \bar{v}_w)^{\frac{1}{2}} - 1] - \bar{\varphi} (1 + \bar{\varphi}^2 \bar{v}_w)^{-\frac{1}{2}} \} = 0 \end{aligned} \quad (29)$$

where

$$(\quad)' \equiv d/d\bar{x}$$

It can easily be shown that in the limit of $v_w \rightarrow 0$ the above equation reduces to that for the impermeable case which comprises part of the analysis found in Reference 9.

D. Final Working Forms of the Governing Equations

By examining the form of the velocity profile given by Equation (22) and the integration required in Equation (5), it is deemed more convenient in defining integrals of the form

$$\Lambda_2^{(\bar{\eta})} \equiv \int_0^{\bar{\eta}} (\bar{u}-1)^2 d\bar{\eta} = \Lambda_2(\bar{\varphi}, \bar{v}_w, \pi, \bar{\eta}) \quad (30)$$

$$\Lambda_1^{(\bar{\eta})} \equiv \int_0^{\bar{\eta}} (\bar{u}-1) d\bar{\eta} = \Lambda_1(\bar{\varphi}, \bar{v}_w, \pi, \bar{\eta}) \quad (31)$$

where it can be readily shown that the term multiplying $d/d\bar{x}(\ell n \bar{R})$, in the latter equation can be replaced with

$$\Lambda_2^{(\bar{\eta}^*)} + (2-\bar{u}^*) \Lambda_1^{(\bar{\eta}^*)} + (1-\bar{u}^*) \bar{\eta}^*$$

Now, by considering the functional dependency of these integral forms on the dependent variables, $\bar{\varphi}$, π , \bar{R} and their implicit dependence on the independent variable, \bar{x} , through the possible streamwise variation of \bar{v} and \bar{u} , then the requisite differentiation of these integral forms when substituted back into Equation (5) will, after collecting terms, produce the result that

$$\begin{aligned} & (\ell n \bar{\varphi})' F_1^{(\bar{\eta}^*)} + (\pi)' F_3^{(\bar{\eta}^*)} + (\ell n \bar{R})' [\Lambda_2^{(\bar{\eta}^*)} + (2-\bar{u}^*) \Lambda_1^{(\bar{\eta}^*)} + \\ & (1-\bar{u}^*) \bar{\eta}^*] + (\ell n \bar{u}_e)' \{ [2\Lambda_2^{(\bar{\eta}^*)} + (4-\bar{u}^*) \Lambda_1^{(\bar{\eta}^*)} + (1-\bar{u}^*) \bar{\eta}^*] - \\ & F_2^{(\bar{\eta}^*)} \} + (\ell n \bar{v}_w)' F_2^{(\bar{\eta}^*)} + \left(\frac{\bar{v}_w}{\bar{R}} \right) - \left(\frac{1}{\bar{R}} \right) [\bar{\tau}^* - \frac{1}{\bar{\varphi}^2}] = 0 \end{aligned} \quad (32)$$

where $\bar{v}_w \equiv \bar{v}_w / \bar{u}_{e,0}$

Noting that the auxiliary equation described above will yield identically the so-called momentum-integral equation if the integration is performed up to $\bar{\eta} = 1$ (with \bar{u}^* and $\bar{\tau}^*$ equal to one and zero respectively at this point), then it can be shown, formally at least, that the last and final governing equation can be written as

$$\begin{aligned}
& (\ln \bar{\varphi})' F_1^{(1)} + (\pi)' F_3^{(1)} + (\ln \bar{R})' [\Lambda_2^{(1)} + \Lambda_1^{(1)}] \\
& + (\ln \bar{U}_e)' [2\Lambda_2^{(1)} + 3\Lambda_1^{(1)} - F_2^{(1)}] + (\ln \bar{V}_w)' F_2^{(1)} \\
& + (\bar{v}_w/\bar{R}) + (1/\bar{R}\bar{\varphi}^2) = 0
\end{aligned} \tag{33}$$

where the superscript (1) indicates that the functions are to be evaluated at $\bar{\eta} = 1$. Also, in terms of the definitions, Λ_1 and Λ_2 , the normalized momentum and displacement thickness are respectively

$$\bar{\Sigma} \equiv \bar{\theta}/\delta = - [\Lambda_2^{(1)} + \Lambda_1^{(1)}] \tag{34}$$

$$\bar{\Omega} \equiv \bar{\delta}^*/\delta = - \Lambda_1^{(1)} \tag{35}$$

The necessary equations to evaluate the terms $F_1^{(\bar{\eta}^*)}$, $F_2^{(\bar{\eta}^*)}$, $F_3^{(\bar{\eta}^*)}$, and the terms $\Lambda_1^{(\bar{\eta}^*)}$, $\Lambda_2^{(\bar{\eta}^*)}$, are given in the Appendix. Also, in lieu of the above equations, the shear stress, $\bar{\tau}^*$, can now be formally represented by

$$\bar{\tau}^* = \beta \bar{\Omega} (\partial G / \partial \bar{\eta}) [A(\bar{\varphi}, \bar{v}_w) + 2B(\bar{v}_w) G(\pi, \bar{\eta})] \tag{36}$$

where

$$(\partial G / \partial \bar{\eta}) = [J(\pi) + K/\bar{\eta}] \tag{37}$$

Finally, Equation (29), (32), and (33) can be recast into the form

$$\begin{vmatrix} A_1 & A_4 & A_7 \\ A_2 & A_5 & A_8 \\ A_3 & A_6 & A_9 \end{vmatrix} \begin{vmatrix} (\ln \bar{\varphi})' \\ (\ln \bar{R})' \\ (\pi)' \end{vmatrix} = \begin{vmatrix} C_1 \\ C_2 \\ C_3 \end{vmatrix} \tag{38}$$

where the coefficients A_i and C_j are also given in the Appendix.

The integration of the system (38) can be readily carried out by means of a high-speed computer, with the development of the boundary layer thus being determined once the external pressure field, the mass-transfer distribution, and the initial conditions are prescribed.

E. Initial and Boundary Conditions

For boundary conditions all that is required is a specification of the external and wall velocity distribution and the unit Reynolds number. The initial conditions which are necessary in order to numerically integrate the above system of equations are the values of $\bar{\varphi}$, π , and \bar{R} at some initial station, \bar{x}_0 .

Usually, integration will start at a station where there is an experimentally determined velocity profile. Then a plot of \bar{u} versus $R\bar{y}$ can be generated from which \bar{R} can be ascertained. Also, by making several judicious guesses on $\bar{\varphi}$, a 'log-law', i.e., Equation (14) which best describes the data can be ascribed. The result is a generalized 'Clauser Plot', as shown in Figure 1 from which the desired value of $\bar{\varphi}$ can be obtained. With $\bar{\varphi}$ and \bar{R} now known, the third parameter, π , is obtained by solving the skin-friction law, Equation (28b), which is recast below in a slightly different form, i.e.,

$$\pi = (k_1/2) \{ [(1 + \bar{\varphi}^2 \bar{v}_w)^{1/2} - 1] (\bar{v}_w \bar{\varphi}/2)^{-1} - k_2 - (1/k_1) \ln(\bar{R} \bar{U}_e / \bar{\varphi}) \} \quad (39)$$

where by definition \bar{U}_e is unity.

If one wished to initiate the solution at a leading edge, the assumption implicit in the analysis are that at $\bar{x}_0 = 0$

$$(\bar{\varphi})_0 = 13.55$$

$$(\pi)_0 = 0.6$$

from which the skin-friction law, yields an expression for \bar{R} , namely

$$\bar{R} = \exp \{ (0.06073 / \bar{v}_w) [(1 + 183.603 \bar{v}_w)^{1/2} - 1] - 0.6104 \} \quad (40)$$

for the leading edge value once a $(\bar{v}_w)_0$ is prescribed.

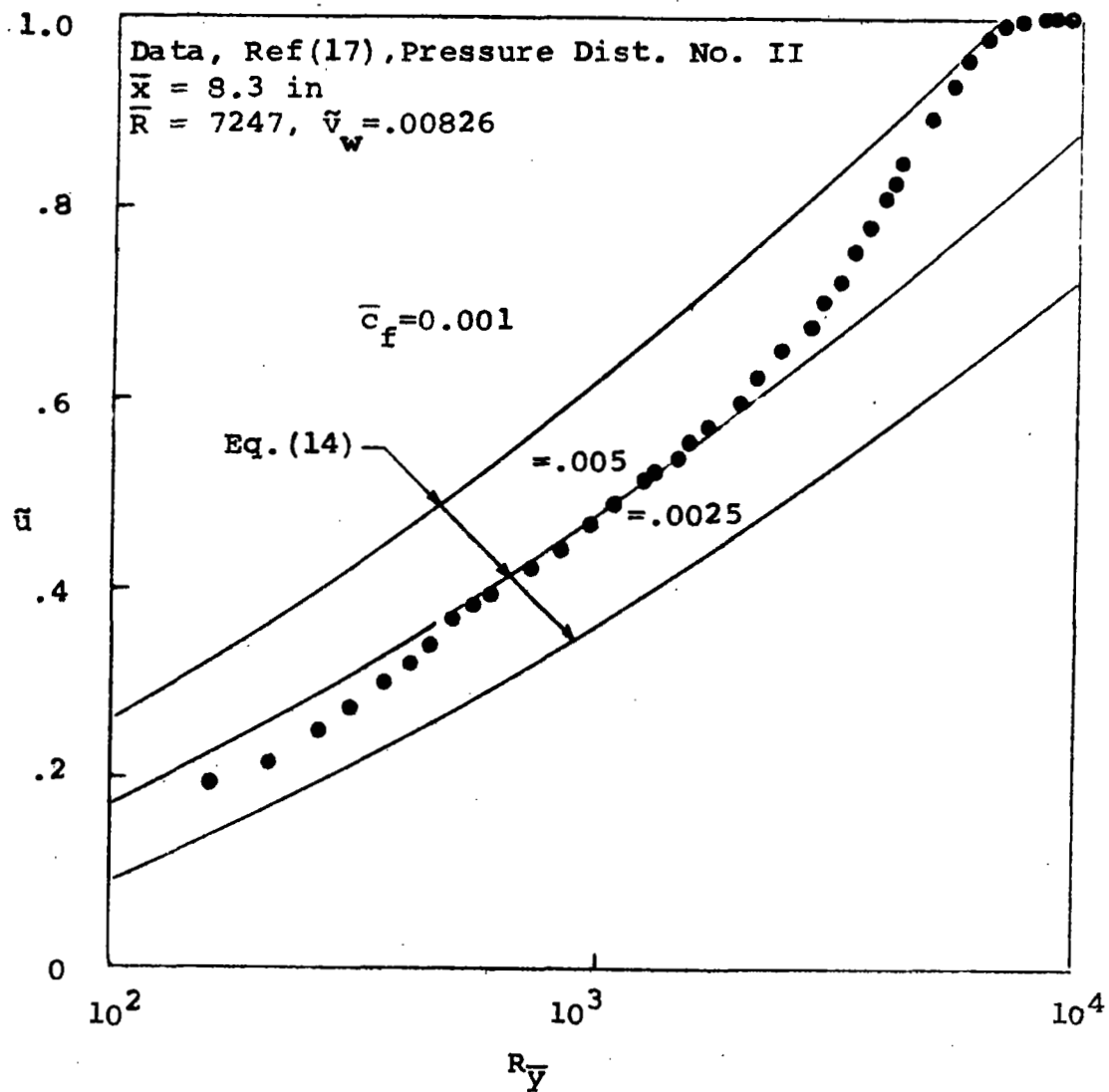


FIGURE 1: TYPICAL "CLAUSER PLOT" REPRESENTATION

RESULTS AND DISCUSSIONS

The numerical results of the analysis of the previous section have been compared with the experimental data of McQuaid (Reference 17). In the three cases considered, the pressure gradient and blowing rate were varied simultaneously but were restricted to air into air injection without heat transfer. Boundary-layer developments were measured with injection rates (\bar{v}_w/\bar{u}_∞) between 0 and 0.0008 at freestream velocities of 50^w and 150^{s,o} ft/sec. At the outset, it must be stated that the skin-friction comparison is not made with the experimental values quoted by McQuaid. It is felt, and as McQuaid concedes, that considerable errors can accrue if use is made of the momentum-integral equation to experimentally infer the wall skin-friction value. Instead, and as McQuaid suggests, the skin-friction coefficient used in all the examples cited is that which has been estimated by means of the Clauser Plot method previously described in Section E. In general, the agreement in all three cases is quite excellent.

McQuaid Pressure Gradient I. - In this case, the boundary layer has developed under a mild adverse pressure gradient and a mild increase in blowing rate. Figure 2 shows the excellent agreement between the analysis and the skin-friction and integral data over the entire range. The ability of the analysis to predict the velocity profiles is also demonstrated in the next figure. In Figure 3, only three profile comparisons are shown; one at the forward portion of the region of interest, one in the center, and one at the end of the measurement domain. Ten such profiles have been measured by McQuaid and all have been compared with the existing analysis showing similar and oftentimes better agreement than those presented herein.

McQuaid Pressure Distribution II. - A mild favorable pressure distribution, together with a decrease in the blowing rate, gives rise to the experimental data reported by McQuaid for this test configuration. Again, the agreement of skin friction, form factor, momentum thickness, and velocity profiles as portrayed in Figures 4 and 5, is quite good.

McQuaid Pressure Distribution III - In this final case, the severest pressure gradient is considered. Over a span of approximately 30 inches the external velocity has increased by a factor of two while the blowing rate has increased three-fold. Again, the comparisons as exemplified in Figures 6 and 7 show rather remarkable agreement. Also, as before, all ten profiles reported have been compared, all showing similar comparative agreement.

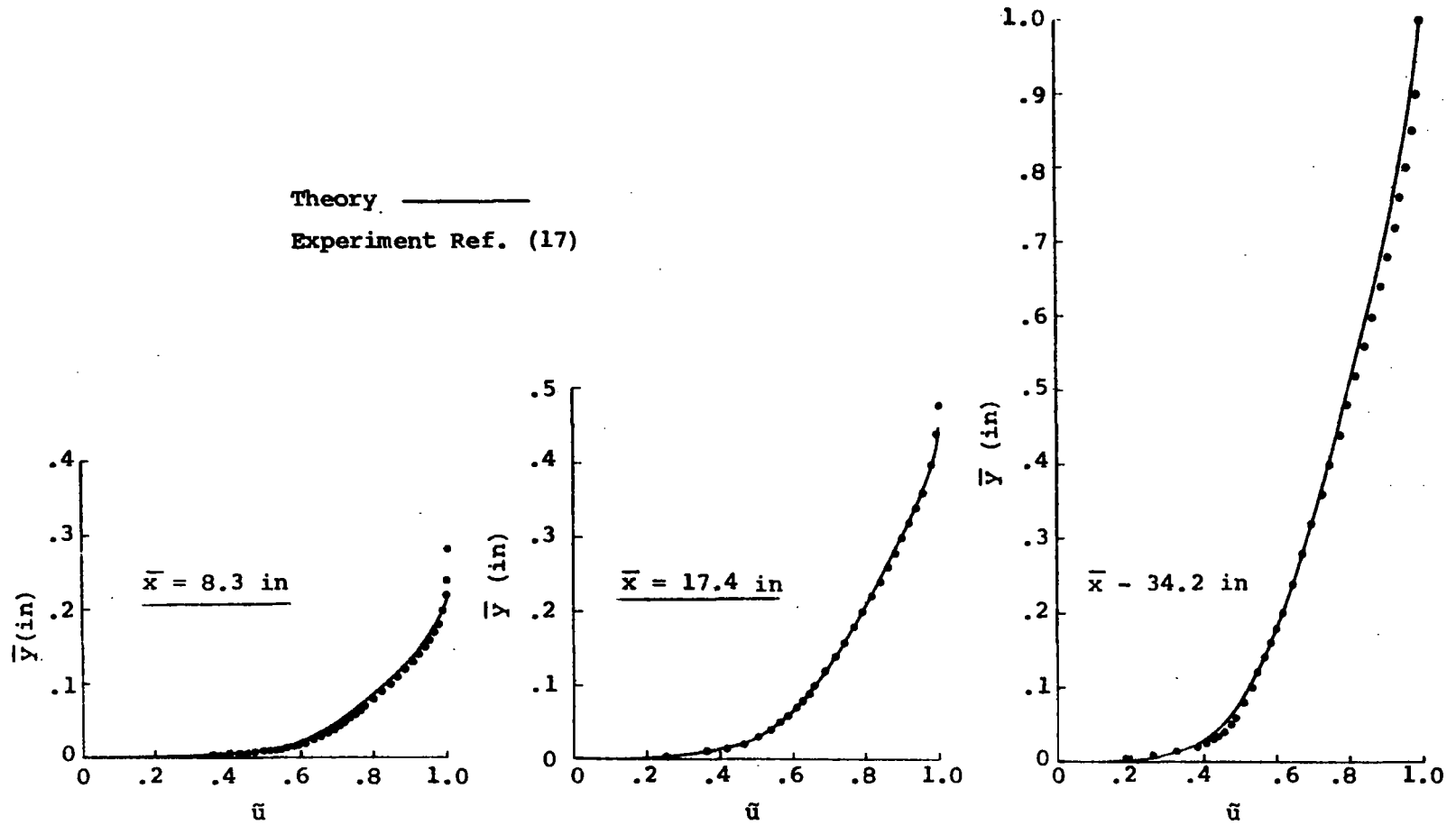


FIGURE 3: TYPICAL VELOCITY PROFILE COMPARISON, MCQUAID DATA, PRESSURE DISTRIBUTION

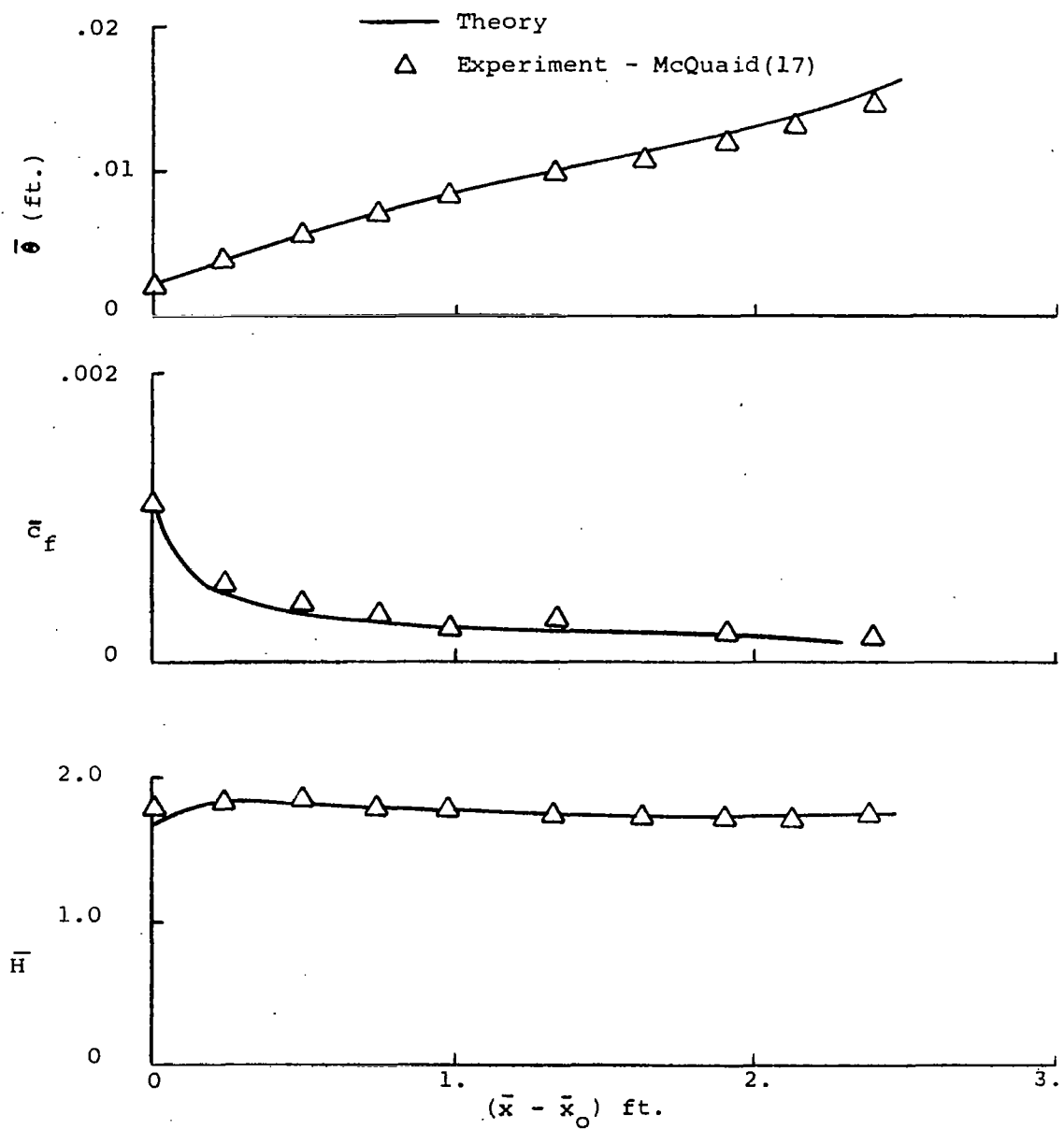


FIGURE 4 - COMPARISON OF THEORY AND EXPERIMENT,
PRESSURE DISTRIBUTION NO. II

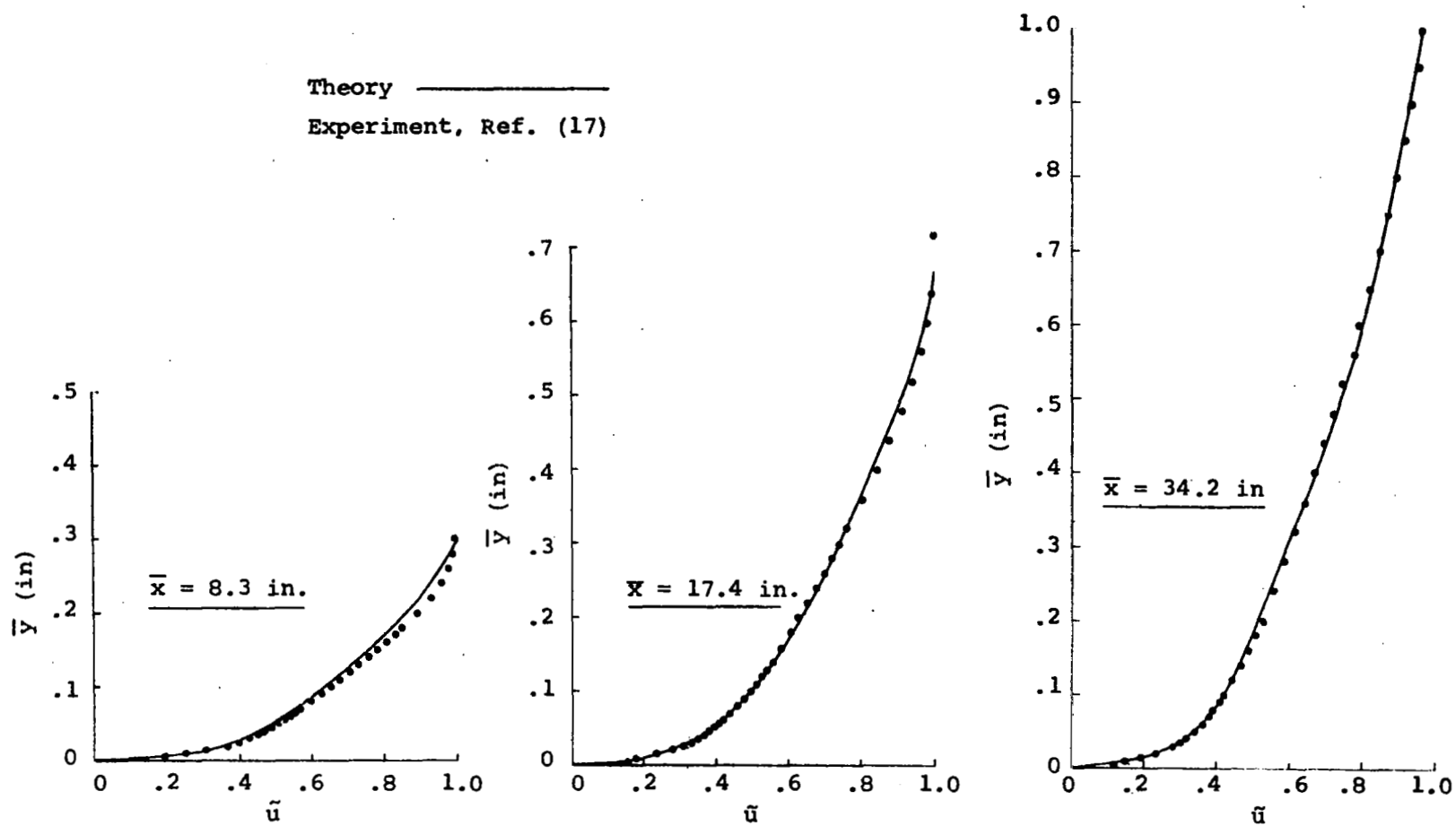


FIGURE 5: TYPICAL VELOCITY PROFILE COMPARISON, MCQUAID DATA, PRESSURE DISTRIBUTION II

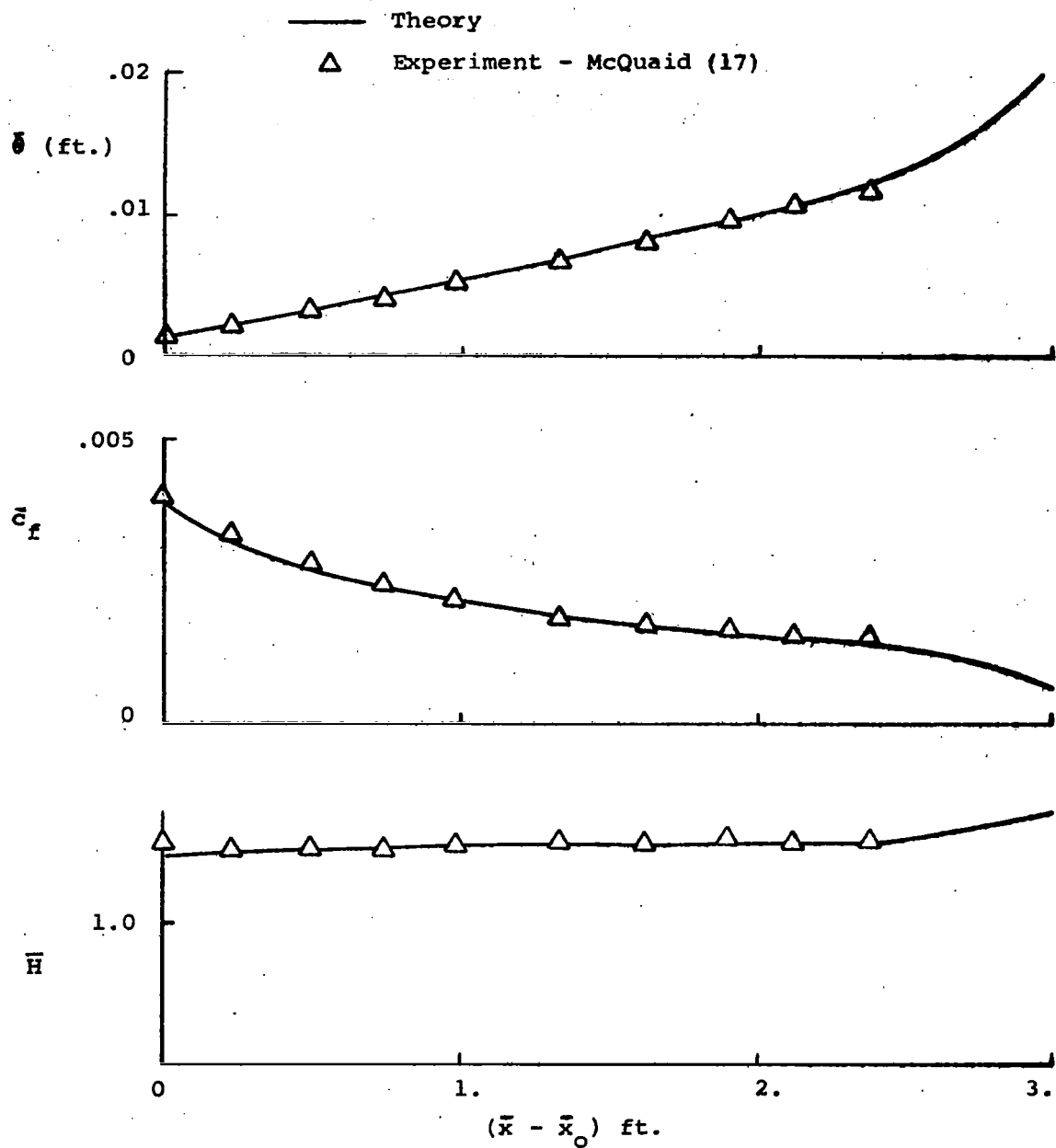


FIGURE 2 - COMPARISON OF THEORY AND EXPERIMENT,
PRESSURE DISTRIBUTION NO.1

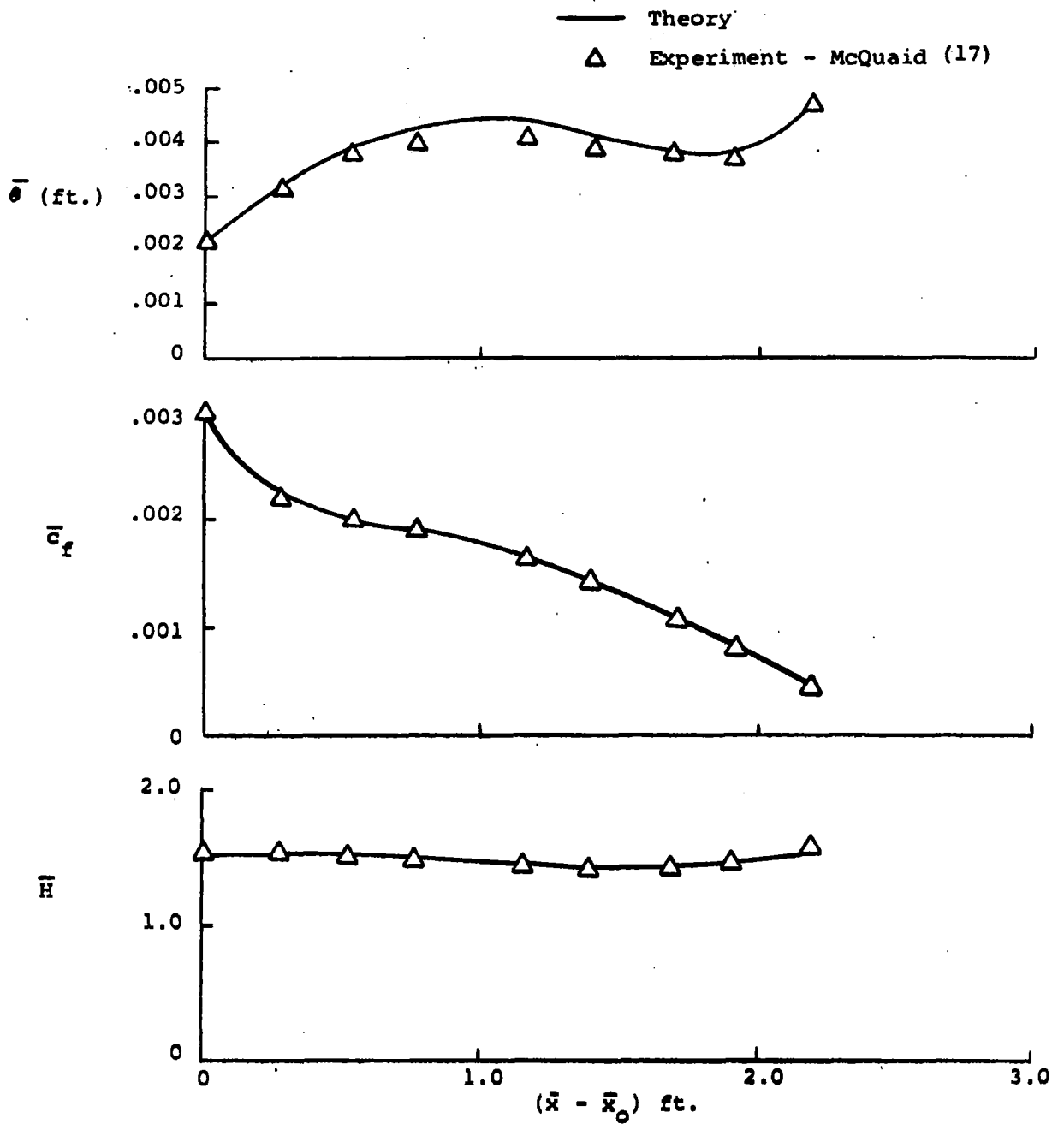


FIGURE 6 - COMPARISON OF THEORY AND EXPERIMENT,
PRESSURE DISTRIBUTION NO. III

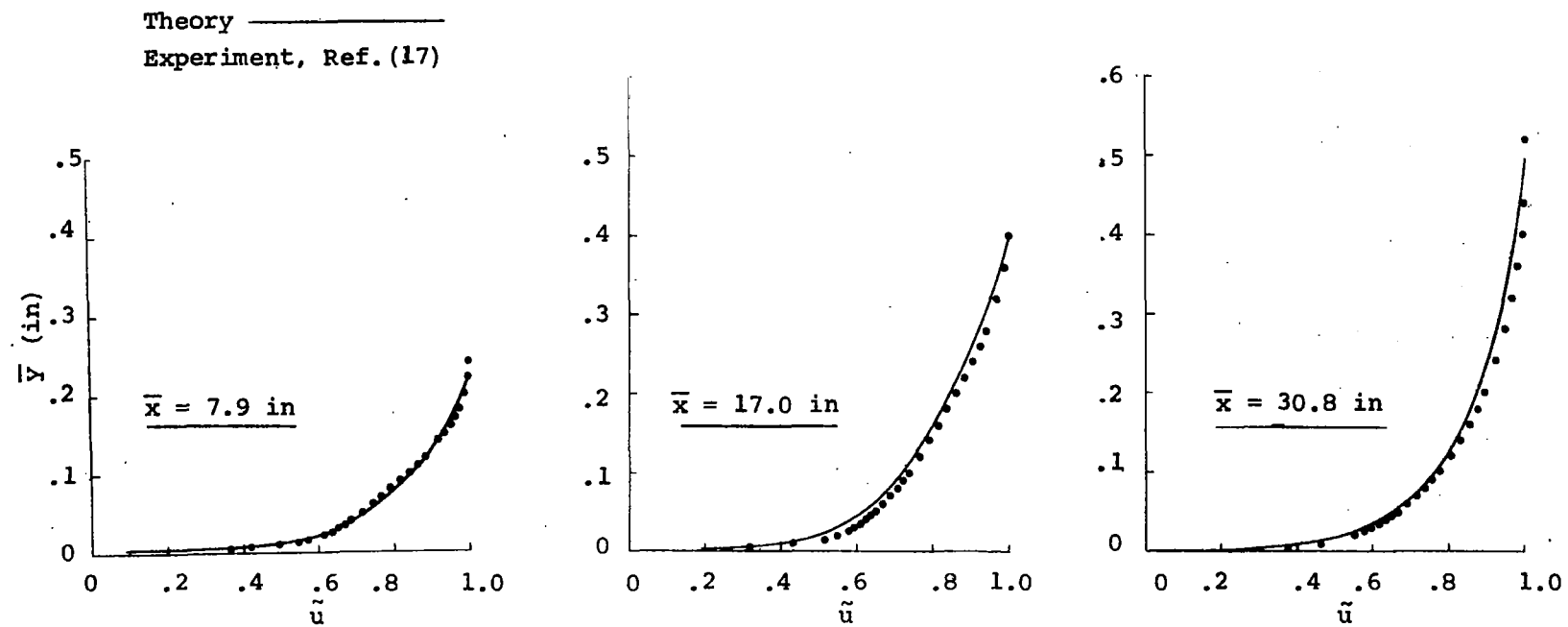


FIGURE 7: TYPICAL VELOCITY PROFILE COMPARISON, MCQUAID DATA, PRESSURE DISTRIBUTION III

CONCLUSIONS

An integral method has been developed for calculating the development of an incompressible boundary layer under the simultaneous influence of both mass transfer and pressure gradient. No need to generate new empirical constants has been required; and also there has been no need to utilize curve fits of non-transpired boundary layer data in the numerical program. The numerical program is quite simple and fast, taking approximately 20 seconds to complete a case during which time ten theoretical velocity profiles can also be generated. No substantial effort has been made to compare this analysis with the integral method of Lubard and Fernandez; suffice to say that both methods show similar comparative tendencies. However, it is felt that the approach present herein is less empirical than that of Lubard and Fernandez, or stated somewhat differently, the empiricism required manifests itself only at a discrete point of the boundary layer, namely, the assumption that Clauser's eddy-viscosity law is valid at only one point, namely at $\bar{\eta}^* = 0.5$. Hence, there is no need to assume gross profile behavior as is required in Reference 15, where

1. a curve fit of Clauser's equilibrium form factor with pressure gradient is required.
2. this same curve fit must be extended to include the transpired problems, and
3. a need to define an equilibrium flow with mass transfer is mandatory.

APPENDIX A

I. The Profile Parameters Λ_1, Λ_2

Considering the velocity profile represented by Equation (22) and, for convenience rewritten here, i.e.,

$$\tilde{u} = 1 + A(\bar{\varphi}, \tilde{v}_w) G(\pi, \bar{\eta}) + B(\tilde{v}_w) G^2(\pi, \bar{\eta}) \quad (\text{A-1})$$

where

$$G(\pi, \bar{\eta}) \equiv [J(\pi)W(\bar{\eta}) + K \ln \bar{\eta}]$$

$$A(\bar{\varphi}, \tilde{v}_w) \equiv - (1 + \bar{\varphi}^2 \tilde{v}_w)^{\frac{1}{2}} (1/\bar{\varphi})$$

$$B(\tilde{v}_w) \equiv (1/4) \tilde{v}_w$$

$$J(\pi) = (1/k_1)(\pi)$$

$$K = - (1/k_1)$$

$$W(\eta) = w(1) - w(\bar{\eta})$$

$$\tilde{u} \equiv \bar{u}/\bar{u}_e, \quad \tilde{v}_w = \bar{v}_w/\bar{u}_e$$

and substituting into the definitions for Λ_1, Λ_2 yields formally two expressions of the form

$$\Lambda_1^{(\bar{\eta})} = A(\bar{\varphi}, \tilde{v}_w) Q_1(\pi, \bar{\eta}) + B(\tilde{v}_w) Q_2(\pi, \bar{\eta}) \quad (\text{A-2})$$

$$\Lambda_2^{(\bar{\eta})} = A^2(\bar{\varphi}, \tilde{v}_w) Q_3(\pi, \bar{\eta}) + 2A(\bar{\varphi}, \tilde{v}_w) B(\tilde{v}_w) Q_4(\pi, \bar{\eta}) + B^2(\tilde{v}_w) Q_5(\pi, \bar{\eta}) \quad (\text{A-3})$$

where the superscript, $(\bar{\eta})$ implies the $\bar{\eta}$ -dependency of the functions brought about by the integration interval $(0, \bar{\eta})$.

The expressions for the Q's involve integrals in the form

$$I(\bar{\eta}, p, q) \equiv \int_0^{\bar{\eta}} [W(\bar{\eta})]^p (\ln \bar{\eta})^q d\bar{\eta} \quad (A-4)$$

where the powers p and q represent integers ranging from 0 to 4. Accordingly it can be shown that

$$Q_1(\pi, \bar{\eta}) = J(\pi) I(\bar{\eta}, 1, 0) + KI(\bar{\eta}, 0, 1)$$

$$Q_2(\pi, \bar{\eta}) = J^2(\pi) I(\bar{\eta}, 2, 0) + 2J(\pi) KI(\bar{\eta}, 1, 1) + K^2 I(\bar{\eta}, 0, 2)$$

$$Q_3(\pi, \bar{\eta}) = Q_2(\pi, \bar{\eta}) \quad (A-5)$$

$$Q_4(\pi, \bar{\eta}) = J^3(\pi) I(\bar{\eta}, 3, 0) + 3J^2(\pi) KI(\bar{\eta}, 2, 1) + 3J(\pi) K^2 I(\bar{\eta}, 1, 2) + K^3 I(\bar{\eta}, 0, 3)$$

$$Q_5(\pi, \bar{\eta}) = J^4(\pi) I(\bar{\eta}, 4, 0) + 4J^3(\pi) KI(\bar{\eta}, 3, 0) + 6J^2(\pi) K^2 I(\bar{\eta}, 2, 2) + 4J(\pi) K^3 I(\bar{\eta}, 1, 3) + K^4 I(\bar{\eta}, 0, 4)$$

and specification of a curve fit to the Coles wake function, i.e.,

$$w(\bar{\eta}) = 2(3\bar{\eta}^2 - 2\bar{\eta}^3) \quad (A-6a)$$

or

$$w(\bar{\eta}) = 1 - \cos(\pi \bar{\eta}) \quad (A-6b)$$

or

$$w(\bar{\eta}) = 39\bar{\eta}^3 - 125\bar{\eta}^4 + 183\bar{\eta}^5 - 133\bar{\eta}^6 + 38\bar{\eta}^7 \quad (A-6c)$$

is all that is required to evaluate the Q's for any $\bar{\eta}$.

II. Derivatives of the Profile Parameters

Examining the auxiliary equation in the body of the report, i.e., Equation (5) indicates that these profile parameters must also be differentiated since the term

$$(d/d\bar{x}) \int_0^{\bar{\eta}^*} \bar{u}^2 d\bar{\eta} - \bar{u}^*(d/d\bar{x}) \int_0^{\bar{\eta}^*} \bar{u} d\bar{\eta}$$

can be written as

$$(d/d\bar{x}) \int_0^{\bar{\eta}^*} [(\bar{u}-1)^2 + 2\bar{u}-1] d\bar{\eta} - \bar{u}^*(d/d\bar{x}) \int_0^{\bar{\eta}^*} [(\bar{u}-1)+1] d\bar{\eta}$$

and since $\bar{\eta}^*$ is a constant (taken here to be 0.5) then the original term in the auxiliary equation can be replaced by

$$(d/d\bar{x}) \Lambda_2^{(\bar{\eta}^*)} + (2-\bar{u}^*)(d/d\bar{x}) \Lambda_1^{(\bar{\eta}^*)}$$

Now, considering the functional dependency of the Λ 's on $\bar{\varphi}, \pi, \bar{v}_w$ and that

$$\bar{v}_w(\bar{x}) = \frac{\bar{v}_w}{\bar{u}_e} = \frac{\bar{v}_w}{\bar{u}_{e,o}} \cdot \frac{\bar{u}_{e,o}}{\bar{u}_e} = \frac{\bar{v}_w(\bar{x})}{\bar{U}_e(\bar{x})}$$

Then the above expression when differentiated and common terms collected can be expressed symbolically as

$$\begin{aligned} (d/d\bar{x}) \Lambda_2^{(\bar{\eta}^*)} + (2-\bar{u}^*)(d/d\bar{x}) \Lambda_1^{(\bar{\eta}^*)} = \\ (\ln \bar{\varphi})' (F_1)^{(\bar{\eta}^*)} + (\ln \bar{v}_w)' (F_2)^{(\bar{\eta}^*)} + (\pi)' (F_3)^{(\bar{\eta}^*)} \\ - (\ln \bar{U}_e)' (F_2)^{(\bar{\eta}^*)} \end{aligned} \quad (A-7)$$

where the expressions for the F's are

$$F_1^{(\bar{\eta}^*)} = (dA/d\bar{\varphi}) (\bar{\varphi}) [2AQ_3 + 2BQ_4 + (2-\tilde{u})^* Q_1] \quad (A-8)$$

$$F_2^{(\bar{\eta}^*)} = (dA/d\tilde{v}_w) (\tilde{v}_w) [2AQ_3 + 2BQ_4 + (2-\tilde{u})^* Q_1] + \\ (dB/d\tilde{v}_w) (\tilde{v}_w) [2AQ_4 + 2BQ_5 + (2-\tilde{u})^* Q_1] \quad (A-9)$$

$$F_3^{(\bar{\eta}^*)} = A^2 (dQ_3/d\pi) + 2AB (dQ_4/d\pi) + B^2 (dQ_5/d\pi) + \\ (2-\tilde{u})^* [A (dQ_1/d\pi) + B (dQ_2/d\pi)] \quad (A-10)$$

Note, that for convenience, the indication of dependency of the various functions on the governing variables has been dropped here. Furthermore, it can readily be shown that

$$(dQ_1/d\pi) = (dJ/d\pi) I(\bar{\eta}, 1, 0) = (1/k_1) I(\bar{\eta}, 1, 0) \\ (dQ_2/d\pi) = (dJ/d\pi) [2JI(\bar{\eta}, 2, 0) + 2KI(\bar{\eta}, 1, 1)] \\ (dQ_3/d\pi) = (dQ_2/d\pi) \quad (A-11) \\ (dQ_4/d\pi) = (dJ/d\pi) [3J^2 I(\bar{\eta}, 3, 0) + 6JKI(\bar{\eta}, 2, 1) + 3K^2 I(\bar{\eta}, 1, 2)] \\ (dQ_5/d\pi) = (dJ/d\pi) [4J^3 I(\bar{\eta}, 4, 0) + 12J^2 KI(\bar{\eta}, 3, 1) + 12JK^2 I(\bar{\eta}, 2, 2) \\ + 4K^3 I(\bar{\eta}, 1, 3)]$$

Likewise the various derivatives of A are

$$\begin{aligned} (dA/d\bar{\varphi}) &= (1/\bar{\varphi}^2) (1+\bar{\varphi}^2 \tilde{v}_w)^{-1/2} \\ (dA/d\tilde{v}_w) &= -(1/2) (\bar{\varphi}) (1+\bar{\varphi}^2 \tilde{v}_w)^{-1/2} \end{aligned} \tag{A-12}$$

and for B all that is required is

$$(dB/d\tilde{v}_w) = (1/4)$$

III Elements of System (38)

$$A_1 = \{ -(1/k_1) + (2/\bar{\varphi} \tilde{v}_w) [(1 + \bar{\varphi}^2 \tilde{v}_w)^{\frac{1}{2}} - 1] - 2\varphi(1 + \bar{\varphi}^2 \tilde{v}_w)^{-\frac{1}{2}} \}$$

$$A_4 = \{1/k_1\}$$

$$A_7 = \{2/k_1\}$$

$$C_1 = - \left\{ (\ln \bar{u}_e)' (1/k_1) + (\ln \tilde{v}_w)' \{ (2/\bar{\varphi} \tilde{v}_w) [(1 + \bar{\varphi}^2 \tilde{v}_w)^{\frac{1}{2}} - 1] - \bar{\varphi}(1 + \bar{\varphi}^2 \tilde{v}_w)^{-\frac{1}{2}} \} \right\}$$

Note: Since $\tilde{v}_w = \bar{v}_w/\bar{u}_e = (\bar{v}_w/\bar{u}_{e,o})(\bar{u}_{e,o}/\bar{u}_e) = (\bar{v}_w/\bar{u}_e)$

$$A_2 = F_1^{(\bar{\eta}^*)}$$

$$A_5 = \Lambda_2^{(\bar{\eta}^*)} + (2-\tilde{u})^* \Lambda_1^{(\bar{\eta}^*)} + (1-\tilde{u})^* \bar{\eta}^*$$

$$A_8 = F_3^{(\bar{\eta}^*)}$$

$$C_2 = - \left\{ (\ln \bar{u}_e)' \{ [2\Lambda_2^{(\bar{\eta}^*)} + (4-\tilde{u})^* \Lambda_1^{(\bar{\eta}^*)} + (1-\tilde{u})^* \bar{\eta}^*] - F_2^{(\bar{\eta}^*)} \} + (\ln \bar{v}_w)' \{ F_2^{(\bar{\eta}^*)} \} + \frac{\tilde{v}_w \tilde{u}^*}{\bar{R}} - \frac{1}{\bar{R}} \{ \tilde{\tau}^* - 1/\varphi^2 \} \right\}$$

$$A_3 = F_1^{(1)}$$

$$A_6 = \Lambda_2^{(1)} + \Lambda_1^{(1)}$$

$$A_9 = F_3^{(1)}$$

$$C_3 = - \left\{ (\ln \bar{u}_e)' \{ 2\Lambda_2^{(1)} + 3\Lambda_1^{(1)} - F_2^{(1)} \} + (\ln \bar{v}_w)' \{ F_2^{(1)} \} + (\tilde{v}_w/\bar{R}) + (1/\bar{R}) \bar{\varphi}^2 \right\}$$

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